

LEFT-RIGHT NONCOMMUTATIVE POISSON ALGEBRAS

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ABSTRACT. The notions of left-right noncommutative Poisson algebra (NP^{lr} -algebra) and left-right algebra with bracket AWB^{lr} are introduced. These algebras are special cases of NLP-algebras and algebras with bracket AWB, respectively, studied earlier. An NP^{lr} -algebra is a noncommutative analogue of the classical Poisson algebra. Properties of the new algebras are studied. The constructions of free objects in the corresponding categories are given. The relations between the properties of NP^{lr} -algebras, the underlying AWB^{lr} , associative and Leibniz algebras are investigated. In the categories AWB^{lr} and NP^{lr} -algebras the notions of actions, representations, centers, actors and crossed modules are described as special cases of the corresponding well-known notions in categories of groups with operations. The cohomologies of NP^{lr} -algebras and AWB^{lr} (resp. of NP^r -algebras and AWB^r) are defined and the relations between them and the Hochschild, Quillen and Leibniz cohomologies are detected. The cases P is a free NP^r or NP^l -algebra, the Hochschild or/and Leibniz cohomological dimension of P is $\leq n$ are considered separately, exhibiting interesting possibilities of representations of the new cohomologies by the well-known ones and relations between the corresponding cohomological dimensions.

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1. INTRODUCTION

In [4] are defined and studied noncommutative Leibniz-Poisson algebras, denoted as NLP-algebras. These are associative algebras P , generally noncommutative, over a ring \mathbb{K} with unit, with bracket operation, according to which they are Leibniz algebras over \mathbb{K} and the Poisson identity holds

$$[a \cdot b, c] = a \cdot [b, c] + [a, c] \cdot b \quad (1.1)$$

for all $a, b, c \in P$. In this paper this identity will be called left Poisson identity and the above defined algebra left noncommutative Poisson algebra, shortly left NP-algebra or NP^l -algebra. It is natural to consider right NP-algebras over a ring \mathbb{K} (NP^r in what follows), which are defined in analogous way satisfying the right Poisson identity

$$[a, b \cdot c] = b \cdot [a, c] + [a, b] \cdot c \quad (1.2)$$

for all $a, b, c \in P$.

A left-right NP-algebra (NP^{lr}) over a ring \mathbb{K} is an algebra, which is an associative and Leibniz algebra and satisfies both (1.1) and (1.2) identities; it is a noncommutative analogue of the classical Poisson algebra. In the same way, an algebra with bracket AWB defined in [9], see below Definition 2.1, is a left AWB, which will be denoted by AWB^l . Obviously, we can define in analogous ways AWB^r and AWB^{lr} as well. Thus we obtain the following commutative diagram of the corresponding categories and inclusion functors

$$\begin{array}{ccccc} \text{AWB}^r & \longleftarrow & \text{AWB}^{lr} & \hookrightarrow & \text{AWB}^l \\ \uparrow & & \uparrow & & \uparrow \\ \text{NP}^r & \longleftarrow & \text{NP}^{lr} & \hookrightarrow & \text{NP}^l. \end{array} \quad (1.3)$$

The purpose of this paper is to study properties of the above defined algebras, including the construction of appropriate complexes for the definition of cohomology, to investigate and to establish relations between them and with the properties of the underlying associative and Leibniz algebras and the corresponding Hochschild [16], Quillen [32] and Leibniz cohomologies [26]. We will see that left-right NP-algebras do not inherit all the properties of left or right NP-algebras. But nevertheless they have interesting intersections and relations with each other due to the specific way of construction of cohomology complexes. We will often omit the proofs, which are analogous to those given in the cases of AWB^l and NP^l -algebras in [9] and [4], respectively.

In Section 2 we present the definitions of new algebras considered in the paper and examples. For convenience of the reader we include the definition of category of interest and some examples as well. In Section 3 we construct free NP^{lr} and NP^r -algebras (resp. AWB^{lr} and AWB^r). The constructions of free NP^l -algebras and AWB^l were given in [4] and [9], respectively. The properties of free objects are investigated, in particular, it is proved that if P is a free NP^l -algebra, then the underlying associative and Leibniz algebras of P are free as well (cf. [4]). In Section 4 we describe action conditions, we present definitions of derivation, extension, crossed module and representation in the categories of the new defined algebras. All these are special cases of the well-known definitions in categories of groups with operations. It turned out that the category of NP^{lr} -algebras is a category of interest, from which, applying the general result of [29], we conclude that this category is

action accessible in the sense of [3]. We construct the universal strict general actor $\text{USGA}(A)$ of an NP^{lr} -algebra A , defined in [8] in a category of interest; we describe center and define actor of NP^{lr} -algebras and, as a special case of the result in [8], we obtain the necessary and sufficient conditions for the existence of an actor of A in terms of $\text{USGA}(A)$. We plan to consider the problem of the existence of an actor in NP^{lr} , or to find individual objects in this category with actor. According to [2] this problem in categories of interest is equivalent to the amalgamation property for protosplit monomorphisms. Here in NP^{lr} we determine the full subcategory of commutative von Neumann regular rings with trivial bracket operations; by the result of [2] we have that in this category always exists an actor for any algebra, and moreover, on the base of the result of the same paper and [10] we conclude that in NP^{lr} there exists a subcategory which satisfies the amalgamation property. This result can be applied to the characterization of effective codescent morphisms in this subcategory. In Section 5 we construct complexes and define the corresponding cohomologies $H_{\text{NP}^{lr}}^n(P, M)$, $H_{\text{AWB}^{lr}}^n(P, M)$, where $P \in \text{NP}^{lr}$ ($P \in \text{AWB}^{lr}$, respectively), and M denotes the corresponding representations of P . In what follows under NP-algebras we will mean NP^r , NP^l and NP^{lr} -algebras, and under AWB we will mean AWB^l , AWB^r and AWB^{lr} . We investigate the relation of the second cohomology with extensions. Like in the case of AWB^l [9], we obtain the isomorphism $H_{\text{AWB}^r}^{n+1}(P, M) \approx H_{\text{Q}}^n(P, M)$ with the Quillen cohomology. From the constructions of the cohomology complexes we detect short exact sequences, from which follow long exact sequences involving cohomologies, relating NP, AWB, Hochschild and Leibniz cohomologies with each other. The special cases, where P is a free NP^l or NP^r -algebra, the Leibniz cohomological dimension or/and the Hochschild cohomological dimension of P is $n / \leq n$ give interesting results, in particular, in these cases we can represent the new cohomologies by the well-known ones and estimate cohomological dimensions of the corresponding AWB and NP-algebras. Note that an operadic approach to similar kind of investigations would be interesting, see e. g. [12, 14, 17, 28]. The cohomology of classical Poisson algebras is defined and studied by J. Huebschmann [18]. Different types of noncommutative Poisson algebras were studied in [20, 21, 34, 33].

2. PRELIMINARY DEFINITIONS AND EXAMPLES

Let \mathbb{K} be a commutative ring with unit. We recall that a *Leibniz algebra* [22, 23] A over \mathbb{K} is a \mathbb{K} -module A equipped with a \mathbb{K} -module homomorphism $[-, -]: A \otimes A \rightarrow A$, called a square bracket, satisfying the Leibniz identity

$$[a, [b, c]] = [[a, b], c] - [[a, c], b], \quad (2.1)$$

for all $a, b, c \in A$. Here and in what follows \otimes means $\otimes_{\mathbb{K}}$.

Definition 2.1.

- (i) A left (resp. right) algebra with bracket over \mathbb{K} , for short, AWB^l (resp. AWB^r), is an associative algebra A equipped with a \mathbb{K} -module homomorphism $[-, -]: A \otimes A \rightarrow A$, such that (1.1) (resp.(1.2)) identity holds.
- (ii) A left-right algebra with bracket over \mathbb{K} (for short, AWB^{lr}) is an associative algebra A equipped with a \mathbb{K} -module homomorphism $[-, -]: A \otimes A \rightarrow A$, such that (1.1) and (1.2) identities hold.

As we have noted in the introduction \mathbf{AWB}^l is the same as algebra with bracket \mathbf{AWB} defined in [9] and \mathbf{NP}^l -algebra is \mathbf{NLP} -algebra defined in [4]. Morphisms between the above defined algebras are \mathbb{K} -module homomorphisms preserving the dot and bracket operations. The corresponding categories will be denoted by \mathbf{NP}^l , \mathbf{NP}^r , \mathbf{NP}^{lr} , \mathbf{AWB}^l , \mathbf{AWB}^r and \mathbf{AWB}^{lr} . The sign “.” of the dot operation will be often omitted, when it is clear from the context, which operation is meant between the elements, e.g. $a \cdot b$ will be written as ab .

Example 2.2.

- (1) Every Poisson algebra is an \mathbf{NP}^{lr} -algebra.
- (2) Any Leibniz algebra A is an \mathbf{NP}^{lr} -algebra with trivial dot operation, i.e. $ab = 0, a, b \in A$.
- (3) Any associative algebra A is an \mathbf{NP}^{lr} -algebra with the usual bracket $[a, b] = ab - ba, a, b \in A$.
- (4) Let A be an associative algebra and let $D: A \rightarrow A$ be a square zero derivation, i.e. $D^2 = 0$ and $D(ab) = (Da)b + a(Db)$. Define the bracket operation by $[a, b] = a(Db) - (Db)a$. It is easy to check that with this bracket operation A is an \mathbf{NP}^l -algebra, but not \mathbf{NP}^r -algebra.
- (5) Let A be an associative algebra of the case (4), where the bracket operation is defined by $[a, b] = (Da)b - b(Da)$. Then A is \mathbf{NP}^r -algebra, but not \mathbf{NP}^l -algebra.
- (6) Let A be an associative algebra with the property that $abc = bac = acb$, for any a, b and $c \in A$, and let $D: A \rightarrow A$ be a square zero derivation. Then A is an \mathbf{NP}^{lr} -algebra with respect to the rule $[a, b] = a(Db) - (Db)a$.
- (7) Every \mathbf{NP} -algebra is an \mathbf{AWB} .
- (8) The following algebra is an \mathbf{AWB}^r (resp. \mathbf{AWB}^l), but not an \mathbf{NP}^r -algebra (resp. \mathbf{NP}^l -algebra). Let A be an associative algebra with a linear application $D: A \rightarrow A$. Then A is an \mathbf{AWB}^r (resp. \mathbf{AWB}^l) where the bracket operation is defined by $[a, b] = (Da)b - b(Da)$ (resp. by $[a, b] = a(Db) - (Db)a$); for the left \mathbf{AWB} this example was given in [9].
- (9) Let A be an associative algebra with a linear application $D: A \rightarrow A$, satisfying the condition $(Da)b - b(Da) = a(Db) - (Db)a$, for any $a, b \in A$. Then the algebra defined in the case (8) is an \mathbf{AWB}^{lr} .
- (10) If the linear application $D: A \rightarrow A$ in the case (9) is a square zero derivation like in case (4), then the algebra with respect to the square bracket $[a, b] = (Da)b - b(Da)$ is an \mathbf{NP}^{lr} -algebra.
- (11) Any associative dialgebra [25] with respect to the operations $ab = a \vdash b$, $[a, b] = a \vdash b - b \dashv a$ (resp. $[a, b] = a \dashv b - b \vdash a$) is an \mathbf{AWB}^r (resp. \mathbf{AWB}^l), but not an \mathbf{AWB}^l (resp. \mathbf{AWB}^r).
- (12) The algebras defined in the case (11) are not generally \mathbf{NP}^r and \mathbf{NP}^l -algebras, respectively. The greatest quotient of these algebras by the congruence relation generated by the relation $[a, [b, c]] \sim [[a, b], c] - [[a, c], b]$, for any a, b and $c \in A$, give examples of \mathbf{NP}^r and \mathbf{NP}^l -algebras, respectively. For \mathbf{NP}^l -algebras this example was given in [4].
- (13) The algebra defined in the case (11), under the additional condition $a \vdash b - b \dashv a = a \dashv b - b \vdash a$, for any $a, b \in A$, is an \mathbf{NP}^{lr} -algebra.
- (14) For an example of a graded version of \mathbf{NP}^l -algebra coming from Physics see [19].

(15) See Section 3 for the constructions of free AWB and NP-algebras.

Definition 2.3. Let $P \in \mathbf{NP}^{lr}$. A subalgebra of P is an associative and Leibniz subalgebra of P . A subalgebra R of P is called a two-sided ideal if $a \cdot r, r \cdot a, [a, r], [r, a] \in R$, for all $a \in P, r \in R$.

The inclusion functor $\text{inc}: \mathbf{Poiss} \rightarrow \mathbf{NP}$ from the category of Poisson algebras to the category of NP-algebras, i.e. left, right or left-right noncommutative Poisson algebras, respectively, has a left adjoint $(-)\text{Poiss}: \mathbf{NP} \rightarrow \mathbf{Poiss}$. This functor assigns to an NP-algebra P the quotient algebra of P with the smallest two-sided ideal spanned by the elements $[x, x]$ and $xy - yx$, for all $x, y \in P$.

Consider the elements $[a, [b, c \cdot d]], [a, [b \cdot c, d]], [a \cdot b, [c, d]]$ and $[a \cdot b, c \cdot d]$ in the category of \mathbf{NP}^{lr} -algebras. The two different decompositions of the first and the fourth elements give the identities

$$[a, c] \cdot [b, d] + [a, c] \cdot [d, b] + [b, c] \cdot [a, d] + [c, b] \cdot [a, d] = 0 \quad (2.2)$$

$$a \cdot c \cdot [b, d] + [a, c] \cdot d \cdot b = c \cdot a \cdot [b, d] + [a, c] \cdot b \cdot d \quad (2.3)$$

The last identity have place in the category of \mathbf{AWB}^{lr} as well.

The two different decompositions of the second and the third elements do not give identities.

Analogously, considering the two different decompositions of the first element in the category of \mathbf{NP}^r -algebras, and the second element in the category of \mathbf{NP}^l -algebras we obtain, respectively, the identities

$$[[a, c] \cdot d, b] = [[a, c], b] \cdot d - [a, c] \cdot [b, d] - [b, c] \cdot [a, d] + c \cdot [[a, d], b] - [c \cdot [a, d], b] \quad (2.4)$$

$$[a, b \cdot [c, d]] + [a, [b, d] \cdot c] = [[a, b \cdot c], d] - [[a, d], b \cdot c]. \quad (2.5)$$

In the categories \mathbf{AWB}^{lr} and \mathbf{NP}^{lr} -algebras we have the following identity as well

$$[a \cdot b, c] - [a, c \cdot b] + [b \cdot c, a] - [b, a \cdot c] + [c \cdot a, b] - [c, b \cdot a] = 0. \quad (2.6)$$

By decomposition of the right side of (2.4) we obtain the identity

$$\begin{aligned} [[a, c] \cdot d, b] &= -[b, [a, c] \cdot d] + [[b, a], c] \cdot d - [[b, c], a] \cdot d - [a, [b, c] \cdot d] + \\ &\quad + [[a, b], c] \cdot d + [[a, d], c \cdot b] - [[a, d], c] \cdot b - [c \cdot [a, d], b]. \end{aligned} \quad (2.7)$$

These identities will be applied in the next Section. The case of \mathbf{NP}^l -algebras was considered in [4].

Recall that an action (a derived action in the sense of [30]) of P on M for associative algebras is given by two \mathbb{K} -module homomorphisms $- \cdot -: P \otimes M \rightarrow M$, $- \cdot -: M \otimes P \rightarrow M$ with the conditions

$$\begin{aligned} p \cdot (m_1 \cdot m_2) &= (p \cdot m_1) \cdot m_2; & m_1 \cdot (p \cdot m_2) &= (m_1 \cdot p) \cdot m_2; \\ (m_1 \cdot m_2) \cdot p &= m_1 \cdot (m_2 \cdot p); & p_1 \cdot (p_2 \cdot m) &= (p_1 \cdot p_2) \cdot m; \\ p_1 \cdot (m \cdot p_2) &= (p_1 \cdot m) \cdot p_2; & m \cdot (p_1 \cdot p_2) &= (m \cdot p_1) \cdot p_2. \end{aligned}$$

An action of P on M for Leibniz algebras is given by two \mathbb{K} -module homomorphisms $[-, -]: P \otimes M \rightarrow M$, $[-, -]: M \otimes P \rightarrow M$ with the conditions

$$\begin{aligned} [p, [m_1, m_2]] &= [[p, m_1], m_2] - [[p, m_2], m_1]; \\ [m_1, [p, m_2]] &= [[m_1, p], m_2] - [[m_1, m_2], p]; \\ [m_1, [m_2, p]] &= [[m_1, m_2], p] - [[m_1, p], m_2]; \\ [p_1, [p_2, m]] &= [[p_1, p_2], m] - [[p_1, m], p_2]; \\ [p_1, [m, p_2]] &= [[p_1, m], p_2] - [[p_1, p_2], m]; \\ [m, [p_1, p_2]] &= [[m, p_1], p_2] - [[m, p_2], p_1]. \end{aligned}$$

Here we recall the definition of category of interest. Let \mathbf{C} be a category of groups with a set of operations Ω and with a set of identities \mathbf{E} , such that \mathbf{E} includes the group identities and the following conditions hold. If Ω_i is the set of i -ary operations in Ω , then:

- (a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;
- (b) the group operations (written additively: $(0, -, +)$) are elements of Ω_0 , Ω_1 and Ω_2 respectively. Let $\Omega'_2 = \Omega_2 \setminus \{+\}$, $\Omega'_1 = \Omega_1 \setminus \{-\}$ and assume that if $*$ $\in \Omega_2$, then Ω'_2 contains $*^\circ$ defined by $x *^\circ y = y * x$. Assume further that $\Omega_0 = \{0\}$;
- (c) for each $*$ $\in \Omega'_2$, \mathbf{E} includes the identity $x * (y + z) = x * y + x * z$;
- (d) for each $\omega \in \Omega'_1$ and $*$ $\in \Omega'_2$, \mathbf{E} includes the identities $\omega(x + y) = \omega(x) + \omega(y)$ and $\omega(x) * y = \omega(x * y)$.

Note that the group operation is denoted additively, but it is not commutative in general. A category \mathbf{C} defined above is called a *category of groups with operations*. The idea of the definition comes from [15] and the axioms are from [30] and [31]. We formulate two more axioms on \mathbf{C} (Axiom (7) and Axiom (8) in [30]).

If C is an object of \mathbf{C} and $x_1, x_2, x_3 \in C$:

Axiom 1.

$$x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1, \text{ for each } * \in \Omega'_2.$$

Axiom 2.

For each ordered pair $(*, \bar{*}) \in \Omega'_2 \times \Omega'_2$ there is a word W such that

$$\begin{aligned} (x_1 * x_2) \bar{*} x_3 &= W(x_1(x_2 x_3), x_1(x_3 x_2), (x_2 x_3) x_1, \\ &\quad (x_3 x_2) x_1, x_2(x_1 x_3), x_2(x_3 x_1), (x_1 x_3) x_2, (x_3 x_1) x_2), \end{aligned}$$

where each juxtaposition represents an operation in Ω'_2 .

A category of groups with operations satisfying Axiom 1 and Axiom 2 is called a *category of interest* in [30].

Denote by \mathbf{E}_G the subset of identities of \mathbf{E} which includes the group laws and the identities (c) and (d). We denote by \mathbf{C}_G the corresponding category of groups with operations. Thus we have $\mathbf{E}_G \hookrightarrow \mathbf{E}$, $\mathbf{C} = (\Omega, \mathbf{E})$, $\mathbf{C}_G = (\Omega, \mathbf{E}_G)$ and there is a full inclusion functor $\mathbf{C} \hookrightarrow \mathbf{C}_G$. The category \mathbf{C}_G is called a *general category of groups with operations* of a category of interest \mathbf{C} (see [5, 8]).

Example 2.4 (Categories of interest). The categories of groups, modules over a ring, vector spaces, associative algebras, associative commutative algebras, Lie algebras, Leibniz algebras are categories of interest. In the example of groups $\Omega'_2 = \emptyset$. In the case of associative algebras with multiplication represented by $*$,

we have $\Omega'_2 = \{*, *^\circ\}$. For Lie algebras take $\Omega'_2 = \{[\ , \], [\ , \]^\circ\}$ (where $[a, b]^\circ = [b, a] = -[a, b]$). For Leibniz algebras, take $\Omega'_2 = \{[\ , \], [\ , \]^\circ\}$, (here $[a, b]^\circ = [b, a]$). The category of alternative algebras is a category of interest as well [30] (see also [7]). The categories of crossed modules and precrossed modules in the category of groups, respectively, are equivalent to categories of interests (see e.g. [8, 6]). According to [2] the category of commutative Von Neumann regular rings is isomorphic to a category of interest. In [29] are given new examples of categories of interest, these are associative dialgebras and associative trialgebras. Dialgebras and trialgebras were defined by Loday [24, 25, 27]. As it is noted in [30] Jordan algebras do not satisfy Axiom 2. It is easy to see that \mathbf{NP}^{lr} is a category of interest; while the categories \mathbf{AWB}^{lr} , \mathbf{AWB}^r , \mathbf{AWB}^l , \mathbf{NP}^r and \mathbf{NP}^l are not categories of interest, they do not satisfy Axiom 2 of the definition.

3. FREE OBJECTS IN \mathbf{NP} AND \mathbf{AWB}

For any set X we shall build a free \mathbf{NP}^{lr} -algebra (resp. \mathbf{NP}^r) over a ring \mathbb{K} . The construction for \mathbf{NP}^l -algebras is given in [4]. Denote by $W(X)$ (resp. $W'(X)$) the set, which contains X and all formal combinations (words) of two operations $(\cdot, [-, -])$ with the elements from X , which have a sense, and which do not contain elements of the forms $[a, [b, c]]$, $[a \cdot b, c]$, $[a, b \cdot c]$ (resp. $[a, [b, c]]$, $[a, b \cdot c]$ and $[[a, c] \cdot d, b]$); moreover, if the element $a \cdot b \cdot [c, d]$ is contained in any word, then the element $b \cdot a \cdot [c, d]$ is not contained in any word, and if the element $[a, b] \cdot [c, d]$ is contained in any word, then $[a, b] \cdot [d, c]$ is not contained in any word, where a, b, c, d are from X or are combinations of elements of X and dot and bracket operations. Let $W_n(X)$ (resp. $W'_n(X)$) be the subset of those words of $W(X)$ (resp. $W'(X)$), which contain n elements of X , i.e. the number of both operations together is $n-1$, $n \geq 1$; we say that this word is of length n . Obviously, $W(X) = \bigcup_{n \geq 1} W_n(X)$ (resp. $W'(X) = \bigcup_{n \geq 1} W'_n(X)$). We define the following maps

$$\sigma_{n,m}, \tau_{n,m} : W_n(X) \times W_m(X) \longrightarrow W_{n+m}(X).$$

$\sigma_{n,m}$ is defined only on those pairs $(a, b) \in W_n(X) \times W_m(X)$, for which the word $a \cdot b \in W_{n+m}(X)$, and by definition $\sigma_{n,m}(a, b) = a \cdot b$, where the right side denotes the word from $W_{n+m}(X)$, which is defined uniquely. Analogously, $\tau_{n,m}$ is defined only on those pairs (a, b) , for which the word $[a, b] \in W_{n+m}(X)$, and by definition $\tau_{n,m}(a, b) = [a, b]$. In the case $a \cdot b, [a, b] \notin W_{n+m}(X)$, $\sigma_{n,m}$ and $\tau_{n,m}$ are not defined. $\sigma'_{n,m}$ and $\tau'_{n,m}$ are defined in analogous ways; it is easy to notice, that $\sigma'_{n,m}$ is defined for any pair $(a, b) \in W'_n(X) \times W'_m(X)$ by $\sigma'_{n,m}(a, b) = a \cdot b$. Let $F(W(X))$ (resp. $F(W'(X))$) be the free \mathbb{K} -module generated by the set $W(X)$ (resp. $W'(X)$). Define the dot operation on $F(W'(X))$ as a linear extension of $\sigma'_{n,m}$ on whole $F(W'(X))$. For those words of $F(W(X))$ (resp. $F(W'(X))$) on which $\sigma_{n,m}$ and $\tau_{n,m}$ are defined (resp. $\tau'_{n,m}$ is defined), we define the dot and the bracket operations (resp. bracket operation) as the linear extensions on $F(W(X))$ of $\sigma_{n,m}$ and $\tau_{n,m}$, respectively (resp. on $F(W'(X))$ of $\tau'_{n,m}$). If the elements $a \cdot b, [a, b] \notin W_{n+m}(X)$, for $a \in W_n(X), b \in W_m(X)$ (resp. $[a, b] \notin W'_{n+m}(X)$, for $a \in W'_n(X), b \in W'_m(X)$), we decompose $a \cdot b$ and $[a, b]$ (resp. $[a, b]$) according to the identities (1.1), (1.2), (2.1), (2.2), (2.3) and the \mathbb{K} -linearity of the dot and the bracket operations (resp. (1.2), (2.1), (2.4) and the \mathbb{K} -linearity of bracket operation), until we obtain the sum of the dot and the bracket operations, respectively (resp. the sum of the bracket

operations), on such pairs of words on which $\sigma_{n,m}$ and $\tau_{n,m}$ are defined (resp. on which $\tau'_{n,m}$ is defined). Acting on every step in such ways we will obtain the sums $d_1 + \dots + d_l$ and $c_1 + \dots + c_k$, respectively, (resp. $c'_1 + \dots + c'_t$), with $b_j, c_i \in F(W_{n+m}(X))$ (resp. $c'_i \in F(W'_{n+m}(X))$) and by definition $(a, b) = d_1 + \dots + d_l$ and $[a, b] = c_1 + \dots + c_k$ (resp. $[a, b] = c'_1 + \dots + c'_t$). Any two different decompositions give the same element of $F(W(X))$ (resp. $F(W'(X))$) and the results of the operations are uniquely defined. By construction $F(W(X))$ (resp. $F(W'(X))$) has a structure of \mathbf{NP}^{lr} -algebra (resp. \mathbf{NP}^r -algebra). Let $i: X \rightarrow F(W(X))$ be the natural injection of sets.

Proposition 3.1. *For any \mathbf{NP}^{lr} -algebra B and a map $\varphi: X \rightarrow B$, there exists a unique homomorphism $\bar{\varphi}: F(W(X)) \rightarrow B$ such that the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{i} & F(W(X)) \\ \varphi \downarrow & \swarrow \bar{\varphi} & \\ B & & \end{array}$$

We omit the proof since it is analogous to the one of [4, Proposition 2.2.1] for \mathbf{NP}^l -algebras. We have for the \mathbf{NP}^r -algebra $F(W'(X))$ the analogous proposition. The constructions of free \mathbf{AWB}^l , \mathbf{AWB}^r and \mathbf{AWB}^{lr} are similar to the construction given above; e.g. in the case of free \mathbf{AWB}^l we take all formal combinations (words) of two operations $(\cdot, [-, -])$ with the elements from X , which have a sense, and do not contain the elements of the form $[a \cdot b, c]$ (cf. with the construction given in [9]).

It is easy to see that the given construction defines the functor \mathbf{F} from the category of sets \mathbf{Set} to \mathbf{NP}^{lr} , where $\mathbf{F}(X) = F(W(X))$, which is a left adjoint to the underlying functor

$$\mathbf{Set} \begin{array}{c} \xrightarrow{\mathbf{F}} \\ \xleftarrow{\mathbf{U}} \end{array} \mathbf{NP}^{lr}.$$

We have between the category \mathbf{Set} and the categories \mathbf{NP}^l [4], \mathbf{NP}^r , \mathbf{AWB}^{lr} , \mathbf{AWB}^l and \mathbf{AWB}^r the analogous pairs of adjoint functors. Let $V_A^{lr}: \mathbf{NP}^{lr} \rightarrow \mathbf{Ass}$, $V_L^{lr}: \mathbf{NP}^{lr} \rightarrow \mathbf{Leib}$ and $T_A^{lr}: \mathbf{AWB}^{lr} \rightarrow \mathbf{Ass}$ be the forgetful functors, where \mathbf{Ass} and \mathbf{Leib} denote the categories of associative and Leibniz algebras, respectively. The analogous meaning will have the symbols $V_A^r, V_A^l, V_L^r, V_L^l, T_A^r, T_A^l$.

Proposition 3.2. *If P is a free \mathbf{NP}^l -algebra, then $V_A^l(P)$ and $V_L^l(P)$ are free associative and free Leibniz algebras, respectively.*

Proof. This theorem in the terminology of NLP-algebras is proved in [4], but the proof for the Leibniz algebras case needs a correction. Therefore we omit the proof of the associative algebra case and present the proof of the second part of the proposition. Let P be the free \mathbf{NP}^l -algebra on the set X . Let X'' be the set of all kind of words of the types $a_1 \dots a_n$ and $a_1 \dots a_n \cdot [\dots]$, where $a_1, \dots, a_n \in X$, $n \geq 1$ and the bracket $[\dots]$ does not contain the words of the forms $[a, [b, c]]$ and $[a, [b, d] \cdot c]$, for $a, b, c, d \in P$. Let $X_2 = X \cup X''$. Applying (1.1), (2.1) and (2.5) it is easy to show that $V_L^l(P)$ is the free Leibniz algebra on the set X_2 . \square

Below we will see that the proof of the analogous statement for \mathbf{NP}^r -algebras is more complicated.

Proposition 3.3. *If P is a free \mathbf{NP}^r -algebra (resp. \mathbf{AWB}^r), then $V_A^r(P)$ and $V_L^r(P)$ (resp. $T_A^r(P)$) are free associative and free Leibniz algebras, respectively (resp. free associative algebra).*

Proof. Let P be the free \mathbf{NP}^r -algebra (resp. \mathbf{AWB}^r) on the set X . Denote by X' the set of all kind of those words of the type $[\dots, \dots]$, which doesn't contain the words of the forms $[a, [b, c]]$, $[a, bc]$, and $[[a, c]d, b]$ (resp. $[a, bc]$). Let $X_1 = X \cup X'$. It is easy to check that $V_A^r(P)$ (resp. $T_A^r(P)$) is a free associative algebra on the set X_1 .

Let X'' be the set of all kind of words of the types $a_1 \cdot \dots \cdot a_n$ and $[\dots, \dots] \cdot a_1 \cdot \dots \cdot a_n$, where $a_1, \dots, a_n \in X$, $n \geq 1$ and the bracket $[\dots, \dots]$ does not contain the words of the form $[a, [b, c]]$, and moreover, for any $a, b, c, d \in P$, the fixed words $[[a, b], c] \cdot d$, $[[a, c], b] \cdot d$, $[[a, b], d] \cdot c$, $[[d, a], b] \cdot c$, $[[d, b], a] \cdot c$, $[[c, a], d] \cdot b$, $[[c, d], a] \cdot b$ and $[[c, d], b] \cdot a$ do not belong to X'' . All other 16 combinations of the type $[[x, y], z] \cdot t$ of the elements a, b, c, d are in X'' . Applying identities (1.2), (2.1) and (2.7) it is easy to check that $V_L^r(P)$ is a free Leibniz algebra on the set $X_2 = X \cup X''$. \square

An analogous statement for \mathbf{AWB}^l is proved in [9]. The following statement proves that left-right NP-algebras do not inherit all properties of left or right NP-algebras.

Proposition 3.4. *If P is a free \mathbf{NP}^{lr} -algebra (resp. \mathbf{AWB}^{lr}), then $V_A^{lr}(P)$ (resp. $T_A^{lr}(P)$) is not a free associative algebra and $V_L^{lr}(P)$ is not a free Leibniz algebra.*

Proof. Let P be the free \mathbf{NP}^{lr} -algebra on the set X . A basis for $V_A^{lr}(P)$ must contain all the elements from X , and all the elements of the form $[a, b]$, where $a, b \in X$. From identity (2.2) or (2.3) it follows that $V_A^{lr}(P)$ is not a free associative algebra. Analogously, from identity (2.3) we see that $T_A^{lr}(P)$ is not a free associative algebra. In the case of the Leibniz algebra $V_L^{lr}(P)$, its basis must contain all the elements from X and all kind of elements of the form $a_1 \cdot \dots \cdot a_n$, where $a_1, \dots, a_n \in X$, $n \geq 1$. The identity (2.6) proves that $V_L^{lr}(P)$ is not a free Leibniz algebra. \square

4. ACTIONS, REPRESENTATIONS AND CROSSED MODULES IN \mathbf{NP} AND \mathbf{AWB}

Under action we will mean a set of actions derived from the corresponding split extension, i.e. a derived action in the sense of [30]. An action for \mathbf{NP}^l -algebras is defined in [4] in the following way:

Definition 4.1 ([4]). Let $M, P \in \mathbf{NP}^l$. We say that P acts on M if we have an action of P on M as associative and Leibniz algebras given respectively by the \mathbb{K} -module homomorphisms

$$- \cdot -: P \otimes M \rightarrow M, \quad - \cdot -: M \otimes P \rightarrow M \quad (4.1)$$

$$[-, -]: P \otimes M \rightarrow M, \quad [-, -]: M \otimes P \rightarrow M \quad (4.2)$$

and the following conditions hold:

$$\begin{aligned} [p_1 \cdot p_2, m] &= p_1 \cdot [p_2, m] + [p_1, m] \cdot p_2; & [p_1 \cdot m, p_2] &= p_1 \cdot [m, p_2] + [p_1, p_2] \cdot m; \\ [m \cdot p_1, p_2] &= m \cdot [p_1, p_2] + [m, p_2] \cdot p_1; & [m_1 \cdot m_2, p] &= m_1 \cdot [m_2, p] + [m_1, p] \cdot m_2; \\ [m_1 \cdot p, m_2] &= m_1 \cdot [p, m_2] + [m_1, m_2] \cdot p; & [p \cdot m_1, m_2] &= p \cdot [m_1, m_2] + [p, m_2] \cdot m_1, \end{aligned}$$

for all $m, m_1, m_2 \in M; p, p_1, p_2 \in P$.

Definition 4.2. Let $M, P \in \mathbf{NP}^r$. We say that P acts on M if we have an action of P on M as associative and Leibniz algebras given by the \mathbb{K} -module homomorphisms (4.1) and (4.2), respectively, and the following conditions hold

$$\begin{aligned} [m, p_1 \cdot p_2] &= p_1 \cdot [m, p_2] + [m, p_1] \cdot p_2; & [p_1, p_2 \cdot m] &= p_2 \cdot [p_1, m] + [p_1, p_2] \cdot m; \\ [p_1, m \cdot p_2] &= m \cdot [p_1, p_2] + [p_1, m] \cdot p_2; & [p, m_1 \cdot m_2] &= m_1 \cdot [p, m_2] + [p, m_1] \cdot m_2; \\ [m_1, m_2 \cdot p] &= m_2 \cdot [m_1, p] + [m_1, m_2] \cdot p; & [m_1, p \cdot m_2] &= p \cdot [m_1, m_2] + [m_1, p] \cdot m_2, \end{aligned}$$

for all $m, m_1, m_2 \in M$ and $p, p_1, p_2 \in P$.

Definition 4.3. Let $M, P \in \mathbf{NP}^{lr}$. We say that P acts on M if we have an action of P on M as left and right NP-algebras.

Actions in the categories \mathbf{AWB}^1 , \mathbf{AWB}^r and \mathbf{AWB}^{lr} are defined in similar ways as in the previous definitions, but obviously, the Leibniz algebra action conditions are not required. If an NP-algebra P acts on M , and M is singular, or equivalently abelian, i.e. $M \cdot M = [M, M] = 0$, then M will be called a *representation* of P . Representation in the category \mathbf{AWB} (for \mathbf{AWB}^1 see [9]) is defined in a similar way. These definitions coincide with the special cases of the general definition of module given in categories of groups with operations in [30]. If M is a representation of P in \mathbf{NP} , then M is a P - P -bimodule, P considered as the underlying associative algebra; analogously, M is an AWB representation of P and a Leibniz representation of P defined in [26]. In the case of Poisson algebras we obtain the representation defined in [13].

A homomorphism between two representations over P is a linear map $f: M \rightarrow M'$ satisfying

$$\begin{aligned} f(p \cdot m) &= p \cdot f(m), & f(m \cdot p) &= f(m) \cdot p, \\ f[p, m] &= [p, f(m)], & f[m, p] &= [f(m), p], \end{aligned}$$

for all $p \in P$ and $m \in M$.

Definition 4.4. Let $P \in \mathbf{NP}$ and M be a representation of P . A derivation from P to M is a linear map $d: P \rightarrow M$ satisfying the conditions

$$\begin{aligned} d(p_1 \cdot p_2) &= d(p_1) \cdot p_2 + p_1 \cdot d(p_2), \\ d[p_1, p_2] &= [d(p_1), p_2] + [p_1, d(p_2)]. \end{aligned}$$

We have the analogous definition for AWB.

Denote by $\text{Der}_{\mathbf{NP}}(P, M)$ the \mathbb{K} -module of such derivations; analogously we will have the notation $\text{Der}_{\mathbf{AWB}}(P, M)$. Any NP-algebra P is a representation of P acting on itself by the operations in P (see [4, Example 2.3.2]). For $p \in P$, the application $\text{ad}_p: P \rightarrow P$ defined by $\text{ad}_p(p') = -[p', p]$ is an example of derivation. The following definition is a special case of the definitions given in [30, 31].

Definition 4.5. Let $P, M \in \mathbf{NP}$. An abelian extension of P by M is a short exact sequence

$$E: 0 \rightarrow M \xrightarrow{i} Q \xrightarrow{j} P \rightarrow 0$$

where $Q \in \mathbf{NP}$ and M is abelian.

Any abelian extension defines on M a unique representation of P in such a way that

$$\begin{aligned} i(j(q) \cdot m) &= q \cdot i(m); & i(m \cdot j(q)) &= i(m) \cdot q; \\ i[j(q), m] &= [q, i(m)]; & i([m, j(q)]) &= [i(m), q]; \end{aligned}$$

for any $m \in M, q \in Q$. Two abelian extensions E and E' of P by M are called *equivalent* if there exists a homomorphism of NP-algebras $f: Q \rightarrow Q'$ inducing the identity morphisms on M and P . Note that in this case f is an isomorphism. Let M be any representation of P . Denote by $\text{Ext}_{\text{NP}}(P, M)$ the set of all equivalence classes of those abelian extensions of P by M , which induce the given representation M of P .

Definition 4.6. Let $M, P \in \mathbf{NP}$ with an action of P on M . A crossed module is a morphism $\mu: M \rightarrow P$ in \mathbf{NP} satisfying the following axioms:

$$\begin{aligned} \mu(p \cdot m) &= p \cdot \mu(m), & \mu(m \cdot p) &= \mu(m) \cdot p, \\ \mu[p, m] &= [p, \mu(m)], & \mu[m, p] &= [\mu(m), p], \\ \mu(m) \cdot m' &= m \cdot m' = m \cdot \mu(m'), & [\mu(m), m'] &= [m, m'] = [m, \mu(m')]. \end{aligned}$$

A homomorphism of crossed modules is a pair $(\phi, \psi): (M, P, \mu) \rightarrow (M', P', \mu')$ where ϕ, ψ are morphisms in \mathbf{NP} such that $\psi\mu = \mu'\phi$ and $\phi(p \cdot m) = \psi(p) \cdot \phi(m)$; $\phi(m \cdot p) = \phi(m) \cdot \psi(p)$; $\phi[p, m] = [\psi(p), \phi(m)]$; $\phi[m, p] = [\phi(m), \psi(p)]$, for all $p \in P, m \in M$.

Examples of representations and crossed modules and the construction of semi-direct products in the category of NP-algebras and AWB are analogous to those given for NP^l -algebras and AWB^l , therefore for these subjects we refer the reader to [4] and [9], respectively.

It is proved in [29] that every category of interest is action accessible in the sense of [3]. Since \mathbf{NP}^{lr} is a category of interest (see Section 2) we obtain

Theorem 4.7. *The category \mathbf{NP}^{lr} is action accessible.*

In [8] for any category of interest \mathbf{C} and for any object $A \in \mathbf{C}$ is defined and constructed the universal strict general actor $\text{USGA}(A)$ of A , which is generally an object of \mathbf{C}_G . Here we give this construction for the category \mathbf{NP}^{lr} . In this case we have three binary operations: the addition, denoted by “+”, the dot and the (square) bracket operations. Ω'_2 from the definition of category of interest is a set with three elements $\Omega'_2 = \{\cdot, [\cdot, \cdot], [\cdot, \cdot]^\circ\}$. Since the addition is commutative, the action corresponding to this operation is trivial. Thus we will deal only with actions, which are defined by dot and bracket operations; the actions of b on a will be denoted as $a \cdot b, b \cdot a, [b, a]$ and $[a, b]$. Below under $*$ operation we will mean either dot or bracket operations. Let $A \in \mathbf{NP}^{lr}$; consider all split extensions of A

$$E_j: 0 \longrightarrow A \xrightarrow{i_j} C_j \xrightarrow{p_j} B_j \longrightarrow 0, \quad j \in \mathbb{J}.$$

Let $\{b_j * \mid b_j \in B_j, * \in \Omega'_2\}$ be the corresponding set of derived actions for $j \in \mathbb{J}$. For any element $b_j \in B_j$ denote $\mathbf{b}_j = \{b_j *, * \in \Omega'_2\}$. Let $\mathbb{B} = \{\mathbf{b}_j \mid b_j \in B_j, j \in \mathbb{J}\}$. Thus each element $\mathbf{b}_j \in \mathbb{B}, j \in \mathbb{J}$, is the special type of a function $\mathbf{b}_j: \Omega'_2 \rightarrow \text{Maps}(A \rightarrow A)$, $\mathbf{b}_j(*) = b_j * -: A \rightarrow A$. According to Axiom 2 of the

definition of a category of interest, we define $*$ operation, $\mathbf{b}_i * \mathbf{b}_k, * \in \Omega'_2$, for the elements of \mathbb{B} by the equalities

$$(\mathbf{b}_i * \mathbf{b}_k) \overline{*}(a) = W(b_i, b_k; a; *, \overline{*}).$$

We define

$$\begin{aligned} (\mathbf{b}_i + \mathbf{b}_k) * (a) &= b_i * a + b_k * a, \\ (-\mathbf{b}_k) * (a) &= -(b_k * a), \\ (-b) * (a) &= -(b * (a)), \\ -(b_1 + \dots + b_n) &= -b_n - \dots - b_1, \end{aligned}$$

where $*$ $\in \Omega'_2$, b, b_1, \dots, b_n are certain combinations of the dot and the bracket operations on the elements of \mathbb{B} , i.e. the elements of the type $\mathbf{b}_{i_1} * \dots * \mathbf{b}_{i_n}$, where $n > 1$. We do not know if the new functions defined by us are again in \mathbb{B} . Denote by $\mathfrak{B}(A)$ the set of functions $(\Omega'_2 \rightarrow \text{Maps}(A \rightarrow A))$ obtained by performing all kinds of the above defined operations on elements of \mathbb{B} and the new obtained elements as results of operations. Let $b \sim b'$ in $\mathfrak{B}(A)$ if $b * a = b' * a$, for any $a \in A, * \in \Omega'_2$. It is an equivalence relation; denote by $\text{USGA}(A)$ be the corresponding quotient algebra. Let \mathbf{NP}_G^{lr} be a general category of groups with operations of the category of interest \mathbf{NP}^{lr} .

Proposition 4.8. *USGA(A) is an object in \mathbf{NP}_G^{lr} .*

Proof. Direct easy checking of the identities. \square

As above, we will write for simplicity $b*(a)$ instead of $(b(*))(a)$, for $b \in \text{USGA}(A)$ and $a \in A$. Define a set of actions of $\text{USGA}(A)$ on A in the following natural way. For $b \in \text{USGA}(A)$ we define $b * a = b * (a)$, $* \in \Omega'_2$. Thus if $b = \mathbf{b}_{i_1} * \dots * \mathbf{b}_{i_n}$, where we mean certain round brackets, we have

$$b \overline{*} a = (\mathbf{b}_{i_1} * \dots * \mathbf{b}_{i_n}) \overline{*}(a).$$

The right side of the equality is defined inductively according to Axiom 2. For $b_k \in B_k, k \in \mathbb{J}$, we have

$$\mathbf{b}_k * a = \mathbf{b}_k * (a) = b_k * a.$$

Also

$$(b_1 + b_2 + \dots + b_n) * a = b_1 * (a) + \dots + b_n * (a).$$

Proposition 4.9. *The set of actions of USGA(A) on A is an action in the category \mathbf{NP}_G^{lr} .*

Proof. It is a special case of the proof of the general statement for categories of interest given in [8]. The checking shows that the set of actions of $\text{USGA}(A)$ on A satisfies conditions of [11, Proposition 1.1], which proves that it is an action in \mathbf{NP}_G^{lr} . \square

Note that this is an action in \mathbf{NP}_G^{lr} , which in general doesn't satisfy the action conditions in \mathbf{NP}^{lr} . Define a map $d: A \rightarrow \text{USGA}(A)$ by $d(a) = \mathbf{a}$, where $\mathbf{a} = \{a, a*, * \in \Omega'_2\}$. Thus we have by definition

$$d(a) * a' = a * a', \quad \forall a, a' \in A, \quad * \in \Omega'_2.$$

The proofs of the following two statements are special cases of those given in [8].

Lemma 4.10. d is a homomorphism in \mathbf{NP}_G^{lr} .

Proposition 4.11. $d: A \longrightarrow \text{USGA}(A)$ is a crossed module in \mathbf{NP}_G^{lr} .

According to the general definition of center [30] (cf. with the definition in [8]) we describe the center of an object in \mathbf{NP}^{lr} as follows

Definition 4.12. The center of $P \in \mathbf{NP}^{lr}$ is

$$Z(P) = \{z \in P \mid z \cdot p = p \cdot z = [z, p] = [p, z] = 0, p \in P\}.$$

It is easy to see that $Z(P) = \text{Ker } d$.

Here we give the definition of an actor in \mathbf{NP}^{lr} (for the case of a category of interest see [8]).

Definition 4.13. For any object A in \mathbf{NP}^{lr} an actor of A is an object $\text{Act}(A) \in \mathbf{NP}^{lr}$, which has an action on A in the same category (i.e. satisfying the conditions of Definition 4.3), such that for any object C in \mathbf{NP}^{lr} with an action on A , there is a unique morphism $\varphi: C \longrightarrow \text{Act}(A)$ with

$$\begin{aligned} c \cdot a &= \varphi(c) \cdot a, & a \cdot c &= a \cdot \varphi(c), \\ [c, a] &= [\varphi(c), a], & [a, c] &= [a, \varphi(c)], \end{aligned}$$

for any $a \in A$ and $c \in C$.

According to the same paper, an actor of A is a split extension classifier for A in the sense of [1]. From the results of [8] we obtain.

Theorem 4.14. For any element $A \in \mathbf{NP}^{lr}$ there exists an actor of A if and only if the semidirect product $\text{USGA}(A) \ltimes A \in \mathbf{NP}^{lr}$. If it is the case, then $\text{Actor}(A) = \text{USGA}(A)$.

At the end of this section we give an example of a subcategory in \mathbf{NP}^{lr} , which satisfies the amalgamation property. This result can be applied to the description of effective codescent morphisms in the corresponding subcategory. For the definition of amalgamation property one can see [2].

Recall that a ring R (generally without a unit) is von Neumann regular if for any $r \in R$ there exists an element $r' \in R$ such that $rr'r = r$.

Proposition 4.15. In the category of \mathbf{NP}^{lr} -algebras there exists a subcategory, which satisfies the amalgamation property.

Proof. Consider the full subcategory in \mathbf{NP}^{lr} , whose objects are commutative von Neumann regular rings with trivial bracket operations. Now it remains to apply the result of [2], where it is proved that the category of (not necessarily unital) commutative von Neumann regular rings satisfies the amalgamation property. \square

5. COHOMOLOGY

We recall the constructions of complexes for Hochschild and Leibniz cohomologies, for cohomologies of left algebras with bracket and left NP-algebras, i.e. AWB^{*l*} and NP^{*l*}-algebras according to [9, 4], respectively. Below for $P \in \mathbf{NP}$ instead of underlying associative and Leibniz algebras $V_A(P), V_L(P)$ and underlying AWB we will write for simplicity just P and will note what kind of algebras we mean, similarly for $P \in \text{AWB}$ and $T_A(P), T_L(P)$.

Let P be a left NP-algebra over a field \mathbb{K} and M a representation of P . In particular, P is an associative algebra and M is a P - P -bimodule and, on the other hand, P is a Leibniz algebra and M is a representation of P in the category of Leibniz algebras. Let $(C_H^*(P, M), \partial_H^n)$ be the Hochschild complex and $(C_L^*(P, M), \partial_L^n)$ be the Leibniz complex. We recall that for $n \geq 0$

$$C_H^n(P, M) = C_L^n(P, M) = \text{Hom}(P^{\otimes n}, M)$$

and coboundary maps ∂_H^n and ∂_L^n are given by

$$\begin{aligned} \partial_H^n(f)(p_1, \dots, p_{n+1}) &= (-1)^{n+1} \left\{ p_1 f(p_2, \dots, p_{n+1}) \right. \\ &\quad \left. + \sum_{i=1}^n (-1)^i f(p_1, \dots, p_i p_{i+1}, \dots, p_{n+1}) + (-1)^{n+1} f(p_1, \dots, p_n) p_{n+1} \right\}, \\ \partial_L^n(f)(p_1, \dots, p_{n+1}) &= [p_1, f(p_2, \dots, p_{n+1})] \\ &\quad + \sum_{i=2}^{n+1} (-1)^i [f(p_1, \dots, \widehat{p}_i, \dots, p_{n+1}), p_i] \\ &\quad + \sum_{1 \leq i < j \leq n+1} (-1)^{j+1} f(p_1, \dots, p_{i-1}, [p_i, p_j], p_{i+1}, \dots, \widehat{p}_j, \dots, p_{n+1}). \end{aligned}$$

Thus $C_H^n(P, M)$ and $C_L^n(P, M)$ are complexes of \mathbb{K} -vector spaces. We will need below the P - P -bimodule M^e , defined by $M^e = \text{Hom}(P, M)$ as a \mathbb{K} -vector space, and a bimodule structure on M^e given by $(p_1 \cdot f)(p_2) = p_1 \cdot f(p_2)$; $(f \cdot p_1)(p_2) = f(p_2) \cdot p_1$. We have an isomorphism of \mathbb{K} -vector spaces $\theta_n: C_H^{n+1}(P, M) \rightarrow C_H^n(P, M^e)$, $n \geq 1$, defined in an obvious way $\theta_n(f)(p_1, \dots, p_n)(p) = f(p_1, \dots, p_n, p)$. Denote the coboundary maps of the complex $C_H^*(P, M^e)$ by $\partial_H^{e,*}$. Let

$$\begin{aligned} \bar{C}_H^*(P, M) &= (C_H^n(P, M), \partial_H^n, n \geq 1), \\ \bar{C}_H^*(P, M^e) &= (C_H^n(P, M^e), \partial_H^{e,n}, n \geq 1), \\ \bar{C}_L^*(P, M) &= (C_L^n(P, M), \partial_L^n, n \geq 1). \end{aligned}$$

Consider the following homomorphisms of cochain complexes, defined in [9] and [4], respectively,

$$\alpha^*: \bar{C}_H^*(P, M) \rightarrow \bar{C}_H^*(P, M^e)$$

and

$$\beta^*: \bar{C}_L^*(P, M) \rightarrow \bar{C}_H^*(P, M^e)$$

and given by

$$\alpha^1(f)(p_1)(p_2) = [p_1, f(p_2)] + [f(p_1), p_2] - f([p_1, p_2]),$$

and for $n > 1$

$$\begin{aligned} \alpha^n(f)(p_1, \dots, p_n)(p_{n+1}) &= [f(p_1, \dots, p_n), p_{n+1}] - f([p_1, p_{n+1}], p_2, \dots, p_n) \\ &\quad - f(p_1, [p_2, p_{n+1}], \dots, p_n) - \dots - f(p_1, \dots, p_{n-1}, [p_n, p_{n+1}]), \\ \beta^{2k+1} &= \theta_{2k+1} \partial_L^{2k+1}, k \geq 0, \\ \beta^{2k} &= \partial_H^{e, 2k-1} \theta_{2k-1}, k \geq 1. \end{aligned}$$

Note that $\alpha^1 = \beta^1$. α^* and β^* are homomorphisms of complexes (see resp. [9] and [4]). Let $\text{cone}(\alpha^*)$ and $\text{cone}(-\beta^*)$ be the mapping cones and $C^*(P, M) =$

$\text{cone}(\alpha^*) \sqcup_{(i_1, i_2)} \text{cone}(-\beta^*)$ the pushout, where i_1 and i_2 are the following injections of complexes

$$\text{cone}(\alpha^*) \xleftarrow{i_1} C_H^{*-1}(P, M^e) \xrightarrow{i_2} \text{cone}(-\beta^*) .$$

Define $C_{\text{NP}^l}^0(P, M) = 0$, $C_{\text{NP}^l}^1(P, M) = \text{Hom}(P, M)$, $C_{\text{NP}^l}^n(P, M) = C^n(P, M)$, $n \geq 2$; $\partial_{\text{NP}^l}^0 = 0$, $\partial_{\text{NP}^l}^1 = (\partial_H^1, 0, \partial_L^1)$, $\partial_{\text{NP}^l}^n = \partial^n$, $n \geq 2$. We have $\partial_{\text{NP}^l}^{n+1} \partial_{\text{NP}^l}^n = 0$, $n \geq 0$, so $\{C_{\text{NP}^l}^n(P, M), \partial_{\text{NP}^l}^n, n \geq 0\}$ is a complex which has the form

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & \text{Hom}(P, M) & & \\
 & \swarrow -\partial_H^1 & \downarrow 0 & \searrow -\partial_L^1 & \\
 C_H^2(P, M) & \oplus & C_H^1(P, M^e) & \oplus & C_L^2(P, M) \\
 \downarrow -\partial_H^2 & \searrow \alpha^2 & \downarrow \partial_H^{e,1} & \swarrow -\beta^2 & \downarrow -\partial_L^2 \\
 C_H^3(P, M) & \oplus & C_H^2(P, M^e) & \oplus & C_L^3(P, M) \\
 \downarrow -\partial_H^3 & \searrow \alpha^3 & \downarrow \partial_H^{e,2} & \swarrow -\beta^3 & \downarrow -\partial_L^3 \\
 \vdots & \vdots & \vdots & \vdots & \vdots
 \end{array}$$

The cohomology vector spaces $H_{\text{NP}^l}^n(P, M)$, $n \geq 0$, of an NP^l -algebra P with coefficients in a representation M of P are defined by

$$H_{\text{NP}^l}^n(P, M) = H^n(C_{\text{NP}^l}^*(P, M), \partial_{\text{NP}^l}^n), \quad n \geq 0.$$

According to [9] the cohomology of AWB is defined by $H_{\text{AWB}}^{n-1}(P, M) = H^n(\text{cone}(\alpha^*))$, for $n \geq 1$, where $P \in \text{AWB}$. Note that in $\text{cone}(\alpha^*)$ the zero term $\text{cone}(\alpha^*)^0$ is zero, and the first one is $C_H^1(P, M) \oplus C_H^0(P, M^e)$. In this paper the cohomology of AWB^l are defined as $H_{\text{AWB}^l}^0(P, M) = 0$ and $H_{\text{AWB}^l}^n(P, M) = H^n(\text{cone}(\alpha^*))$, for $n \geq 1$.

Now we shall define the cohomology vector spaces of NP^r and NP^{lr} -algebras. Let $\theta'_n : C_H^{n+1}(P, M) \rightarrow C_H^n(P, M^e)$, for $n \geq 1$, be the homomorphism defined by $\theta'_1(f)(p_1)(p_2) = f(p_2, p_1)$ and $\theta'_n(f)(p_1, p_2, \dots, p_n)(p_{n+1}) = f(p_{n+1}, p_1, \dots, p_n)$, $n > 1$. It is easy to see that θ'_n is an isomorphism for each $n \geq 1$. Define the homomorphisms

$$\begin{aligned}
 \alpha'^* : \bar{C}_H^*(P, M) &\rightarrow \bar{C}_H^*(P, M^e), \\
 \beta'^* : \bar{C}_L^*(P, M) &\rightarrow \bar{C}_H^*(P, M^e),
 \end{aligned}$$

by

$$\alpha'^1(f)(p_1)(p_2) = [f(p_2), p_1] + [p_2, f(p_1)] - f([p_2, p_1])$$

and for $n > 1$ by

$$\begin{aligned}
 \alpha'^n(f)(p_1, \dots, p_n)(p_{n+1}) &= [p_{n+1}, f(p_1, \dots, p_n)] - f([p_{n+1}, p_1], p_2, \dots, p_n) \\
 &\quad - f(p_1, [p_{n+1}, p_2], \dots, p_n) - \dots - f(p_1, \dots, p_{n-1}, [p_{n+1}, p_n]),
 \end{aligned}$$

$$\beta'^{2k+1} = \theta'_{2k} \partial_L^{2k+1}, \quad k \geq 0, \quad \beta'^{2k} = \partial_H^{e, 2k-1} \theta'_{2k-1}, \quad k \geq 1.$$

We have $\alpha'^1 = \beta'^1$. Easy checking shows that α'^* and β'^* are homomorphisms of complexes.

By taking the pushout $C'^*(P, M) = \text{cone}(\alpha'^*) \sqcup_{(i'_1, i'_2)} \text{cone}(-\beta'^*)$, where i'_1 and i'_2 are the following injections of complexes

$$\text{cone}(\alpha'^*) \xleftarrow{i'_1} C_H^{*-1}(P, M^e) \xrightarrow{i'_2} \text{cone}(-\beta'^*) ,$$

we construct the complex analogous to $\{C_{\text{NP}^l}^n(P, M), \partial_{\text{NP}^l}^n, n \geq 0\}$, which will be denoted by $\{C_{\text{NP}^r}^n(P, M), \partial_{\text{NP}^r}^n, n \geq 0\}$. The complex has the form

$$\begin{array}{ccccc}
& & 0 & & \\
& & \downarrow & & \\
& & \text{Hom}(P, M) & & \\
& \swarrow -\partial_H^1 & \downarrow 0 & \searrow -\partial_L^1 & \\
C_H^2(P, M) & \oplus & C_H^1(P, M^e) & \oplus & C_L^2(P, M) \\
\downarrow -\partial_H^2 & \searrow \alpha'^2 & \downarrow \partial_H^{e,1} & \swarrow -\beta'^2 & \downarrow -\partial_L^2 \\
C_H^3(P, M) & \oplus & C_H^2(P, M^e) & \oplus & C_L^3(P, M) \\
\downarrow -\partial_H^3 & \searrow \alpha'^3 & \downarrow \partial_H^{e,2} & \swarrow -\beta'^3 & \downarrow -\partial_L^3 \\
C_H^4(P, M) & \oplus & C_H^3(P, M^e) & \oplus & C_L^4(P, M) \\
\downarrow -\partial_H^4 & \searrow \alpha'^4 & \downarrow \partial_H^{e,3} & \swarrow -\beta'^4 & \downarrow -\partial_L^4 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

The cohomology vector spaces of an NP^r -algebra P with coefficients in a representation M of P are defined as the cohomologies of this complex and denoted as $H_{\text{NP}^r}^n(P, M), n \geq 0$.

Now we construct the complex for the cohomology of an NP^{lr} -algebra P . Consider the following pairs of homomorphisms of complexes

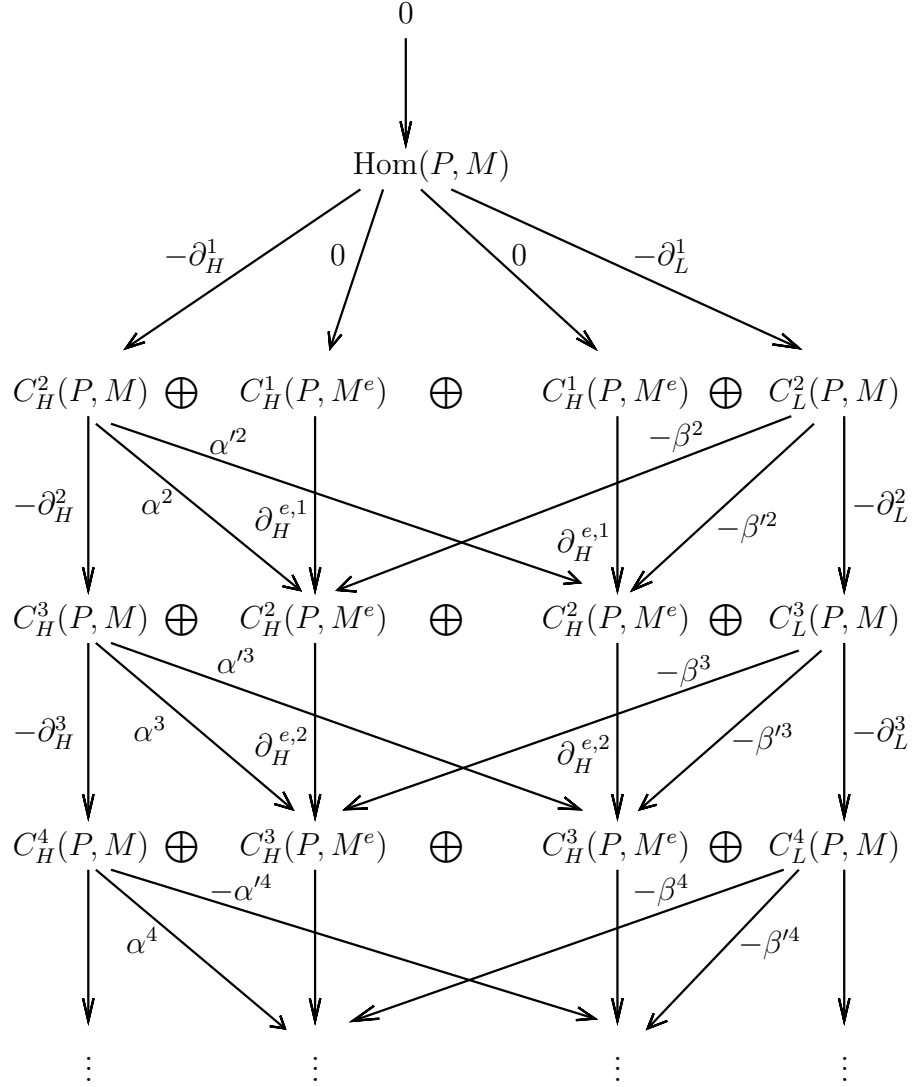
$$\begin{aligned}
(\alpha^*, \alpha'^*) &: \bar{C}_H^*(P, M) \rightarrow \bar{C}_H^*(P, M^e) \oplus \bar{C}_H^*(P, M^e) , \\
(\beta^*, \beta'^*) &: \bar{C}_L^*(P, M) \rightarrow \bar{C}_H^*(P, M^e) \oplus \bar{C}_H^*(P, M^e) .
\end{aligned}$$

From these homomorphisms we obtain two cones: $\text{cone}(\alpha^*, \alpha'^*)$ and $\text{cone}(\beta^*, \beta'^*)$. We have the following homomorphisms of complexes

$$\text{cone}(\alpha^*, \alpha'^*) \xleftarrow{j_1} C_H^{*-1}(P, M^e) \oplus C_H^{*-1}(P, M^e) \xrightarrow{j_2} \text{cone}(-\beta^*, -\beta'^*) .$$

The pushout of the pair (j_1, j_2) gives the desired complex. In particular, we take $C_{\text{NP}^{lr}}^0(P, M) = 0$, $C_{\text{NP}^{lr}}^1(P, M) = \text{Hom}(P, M)$, $C_{\text{NP}^{lr}}^n(P, M) = C_H^n(P, M) \oplus C_H^n(P, M^e) \oplus C_L^n(P, M)$, for $n \geq 2$; $\partial_{\text{NP}^{lr}}^0 = 0$, $\partial_{\text{NP}^{lr}}^1 = (-\partial_H^1, 0, 0, -\partial_L^1)$, $\partial_{\text{NP}^{lr}}^n$ is induced by $\alpha^n, \alpha'^n, \partial_H^{e,n-1}, \partial_H^{e,n-1}, \beta^n, \beta'^n$, for $n \geq 2$.

We have $\partial_{\text{NP}^{lr}}^{n+1} \partial_{\text{NP}^{lr}}^n = 0$, for $n \geq 0$, therefore $\{C_{\text{NP}^{lr}}^n(P, M), \partial_{\text{NP}^{lr}}^n, n \geq 0\}$ is a complex; it has the following form



The cohomology vector spaces of an NP^{lr} -algebra P with coefficients in a representation M of P are defined as the cohomologies of this complex and denoted as $H_{\text{NP}^{lr}}^n(P, M)$, $n \geq 0$.

As in the case of NP^l -algebras in [4], we define restricted second cohomology of NP^{lr} -algebras. We have the natural injection $C_H^2(P, M) \oplus C_L^2(P, M) \rightarrow C_{\text{NP}^{lr}}^2(P, M)$ on to the first and the fourth summands; the image of this injection will be denoted again by the sum $C_H^2(P, M) \oplus C_L^2(P, M)$. Consider the restriction

$$d_{\text{NP}^{lr}}^2 = \partial_{\text{NP}^{lr}}^2|_{C_H^2(\cdot) \oplus C_L^2(\cdot)}.$$

We define the 2-dimensional restricted cohomology of the NP^{lr} -algebra P with coefficients in M by

$$\mathbb{H}_{\text{NP}^{lr}}^2(P, M) = \text{Ker } d_{\text{NP}^{lr}}^2 / \text{Im } \partial_{\text{NP}^{lr}}^1.$$

The obvious injection

$$\kappa: \text{Ker } d_{\text{NP}^{lr}}^2 \longrightarrow \text{Ker } \partial_{\text{NP}^{lr}}^2$$

induces the injection of the corresponding cohomologies

$$\chi: \mathbb{H}_{\text{NP}^{lr}}^2(P, M) \longrightarrow H_{\text{NP}^{lr}}^2(P, M).$$

$\mathbb{H}_{\text{NP}^r}^2(P, M)$ is defined in analogous way as for NP^l -algebras.

The cohomologies of AWB^r and AWB^{lr} are defined by

$$\begin{aligned} H_{\text{AWB}^r}^*(P, M) &= H^*(\text{cone}(\alpha'^*)), \\ H_{\text{AWB}^{lr}}^*(P, M) &= H^*(\text{cone}(\alpha^*, \alpha'^*)). \end{aligned}$$

From the definitions we obtain

Lemma 5.1.

(i) For $P \in \mathbf{NP}$ we have

$$\begin{aligned} H_{\text{NP}}^0(P, M) &= 0, \\ H_{\text{NP}}^1(P, M) &= \text{Der}_{\text{NP}}(P, M). \end{aligned}$$

(ii) For $P \in \mathbf{AWB}$ we have

$$\begin{aligned} H_{\text{AWB}}^0(P, M) &= 0, \\ H_{\text{AWB}}^1(P, M) &= \text{Der}_{\text{AWB}}(P, M), \quad \text{and} \\ H_{\text{AWB}}^2(P, M) &\cong \text{Ext}_{\text{AWB}}(P, M). \end{aligned}$$

Proof. (i) The proof follows directly from the fact that $C_{\text{NP}}^0(P, M) = 0$, from the definition of ∂_{NP}^1 and the definition of a derivation.

(ii) Since the zero term in the corresponding cone complex is zero, the first equality follows from the definition of the cohomology. The proofs of other two equalities of (ii) for AWB^r and AWB^{lr} are similar to the proofs given in [9] for AWB^l . □

Theorem 5.2. $\mathbb{H}_{\text{NP}}^2(P, M) \cong \text{Ext}_{\text{NP}}(P, M)$.

Proof. The proof is similar to the one for NP^l -algebras presented in [4]. □

Corollary 5.3.

(i) If P is a free NP -algebra, then

$$\mathbb{H}_{\text{NP}}^2(P, M) = 0$$

for any representation M of P .

(ii) If P is a free NP^l (resp. NP^r -algebra), then

$$H_{\text{NP}^l}^n(P, M) = 0 \quad (\text{resp. } H_{\text{NP}^r}^n(P, M) = 0),$$

for $n \geq 3$ and any representation M .

Proof. (i) Since for a free NP -algebra P every extension $0 \rightarrow M \xrightarrow{i} Q \xrightarrow{j} P \rightarrow 0$ splits, the fact follows from Theorem 5.2.

(ii) From Proposition 3.2 (resp. Proposition 3.3) it follows that $V_A^l(P)$ (resp. $V_A^r(P)$) is a free associative algebra and $V_L^l(P)$ (resp. $V_L^r(P)$) is a free Leibniz algebra. It is well known that cohomologies of free associative algebras and free Leibniz algebras vanish in dimensions ≥ 2 [26]. Thus we have $H_H^n(P, -) = 0$ and

$H_L^n(P, -) = 0$ for $n \geq 2$. From this and from the fact that α^* and β^* (resp. α'^* and β'^*) are homomorphisms of cochain complexes, by diagram chasing we obtain that $C_{\text{NP}^l}^*(P, M)$ (resp. $C_{\text{NP}^r}^*(P, M)$) is exact in dimensions > 2 for a free NP^l -algebra (resp. NP^r -algebra) P . \square

Lemma 5.4. *If P is a free AWB^r , then*

$$H_{\text{AWB}^r}^n(P, M) = 0,$$

for $n \geq 2$ (according to the notation in [9], $n \geq 1$) and any representation M of P .

The proof is analogous to the proof of this fact for AWB^l given in [9] and therefore it is omitted.

In [9] it is proved that if P is AWB^l , then its cohomologies are isomorphic to Quillen cohomologies. In the similar way, applying Lemma 5.4 we have

Theorem 5.5. $H_{\text{AWB}^r}^{n+1}(P, M) \approx H_Q^n(P, M)$.

From the constructions of the cohomology complexes we obtain the following short exact sequences of complexes

$$0 \rightarrow \text{cone}(\alpha^*) \rightarrow C_{\text{NP}^l}^*(P, M) \rightarrow C_L^*(P, M) \rightarrow 0, \quad * \geq 3 \quad (a_1)$$

$$0 \rightarrow \text{cone}(\alpha'^*) \rightarrow C_{\text{NP}^r}^*(P, M) \rightarrow C_L^*(P, M) \rightarrow 0, \quad * \geq 3 \quad (a_2)$$

$$0 \rightarrow \text{cone}(\alpha^*, \alpha'^*) \rightarrow C_{\text{NP}^{lr}}^*(P, M) \rightarrow C_L^*(P, M) \rightarrow 0, \quad * \geq 3 \quad (a)$$

$$0 \rightarrow \text{cone}(-\beta^*) \rightarrow C_{\text{NP}^l}^*(P, M) \rightarrow C_H^*(P, M) \rightarrow 0, \quad * \geq 3 \quad (b_1)$$

$$0 \rightarrow \text{cone}(-\beta'^*) \rightarrow C_{\text{NP}^r}^*(P, M) \rightarrow C_H^*(P, M) \rightarrow 0, \quad * \geq 3 \quad (b_2)$$

$$0 \rightarrow \text{cone}(-\beta^*, -\beta'^*) \rightarrow C_{\text{NP}^{lr}}^*(P, M) \rightarrow C_H^*(P, M) \rightarrow 0, \quad * \geq 1 \quad (b)$$

$$0 \rightarrow C_H^{*-1}(P, M^e) \rightarrow C_{\text{AWB}^l}^*(P, M) \rightarrow C_H^*(P, M) \rightarrow 0, \quad * \geq 1 \quad (c_1)$$

$$0 \rightarrow C_H^{*-1}(P, M^e) \rightarrow C_{\text{AWB}^r}^*(P, M) \rightarrow C_H^*(P, M) \rightarrow 0, \quad * \geq 1 \quad (c_2)$$

$$0 \rightarrow C_H^{*-1}(P, M^e) \xrightarrow{i_3} C_{\text{AWB}^{lr}}^*(P, M) \rightarrow C_{\text{AWB}^l}^*(P, M) \rightarrow 0, \quad * \geq 1 \quad (c)$$

$$0 \rightarrow C_H^{*-1}(P, M^e) \xrightarrow{i_2} C_{\text{AWB}^{lr}}^*(P, M) \rightarrow C_{\text{AWB}^r}^*(P, M) \rightarrow 0, \quad * \geq 1 \quad (c')$$

$$0 \rightarrow C_H^{*-1}(P, M^e) \rightarrow C_{\text{NP}^l}^*(P, M) \rightarrow C_H^*(P, M) \oplus C_L^*(P, M) \rightarrow 0, \quad * \geq 3 \quad (d_1)$$

$$0 \rightarrow C_H^{*-1}(P, M^e) \rightarrow C_{\text{NP}^r}^*(P, M) \rightarrow C_H^*(P, M) \oplus C_L^*(P, M) \rightarrow 0, \quad * \geq 3 \quad (d_2)$$

$$0 \rightarrow C_H^{*-1}(P, M^e) \xrightarrow{i_3} C_{\text{NP}^{lr}}^*(P, M) \rightarrow C_{\text{NP}^l}^*(P, M) \rightarrow 0, \quad * \geq 3 \quad (d)$$

$$0 \rightarrow C_H^{*-1}(P, M^e) \xrightarrow{i_2} C_{\text{NP}^{lr}}^*(P, M) \rightarrow C_{\text{NP}^r}^*(P, M) \rightarrow 0, \quad * \geq 3 \quad (d')$$

$$0 \rightarrow C_H^{*-1}(P, M^e) \oplus C_H^{*-1}(P, M^e) \rightarrow C_{\text{AWB}^{lr}}^*(P, M) \rightarrow C_H^*(P, M) \rightarrow 0, \quad * \geq 3 \quad (e)$$

$$0 \rightarrow C_H^{*-1}(P, M^e) \oplus C_H^{*-1}(P, M^e) \rightarrow C_{\text{NP}^{lr}}^*(P, M) \rightarrow C_H^*(P, M) \oplus C_L^*(P, M) \rightarrow 0 \quad (f)$$

$$0 \rightarrow C_H^{*-1}(P, M^e) \xrightarrow{(i_2, -i_3)} \text{cone}(\alpha^*) \oplus \text{cone}(-\beta^*) \rightarrow C_{\text{NP}^l}^*(P, M) \rightarrow 0, \quad * \geq 3 \quad (g_1)$$

$$0 \rightarrow C_H^{*-1}(P, M^e) \xrightarrow{(i_2, -i_3)} \text{cone}(\alpha'^*) \oplus \text{cone}(-\beta'^*) \rightarrow C_{\text{NPr}}^*(P, M) \rightarrow 0, \quad * \geq 3$$

(g_2)

$$\begin{aligned} 0 \rightarrow C_H^{*-1}(P, M^e) \oplus C_H^{*-1}(P, M^e) &\xrightarrow{((i_2, i_3), (-i_4, -i_5))} \text{cone}(\alpha^*, \alpha'^*) \oplus \text{cone}(-\beta^*, -\beta'^*) \\ &\rightarrow C_{\text{NP}^{\text{tr}}}^*(P, M) \rightarrow 0, \end{aligned} \tag{g}$$

$$0 \rightarrow C_H^{*-1}(P, M^e) \rightarrow \text{cone}(-\beta^*) \rightarrow C_L^*(P, M) \rightarrow 0, \quad * \geq 3 \quad (h_1)$$

$$0 \longrightarrow C_H^{*-1}(P, M^e) \longrightarrow \text{cone}(-\beta'^*) \longrightarrow C_L^*(P, M) \longrightarrow 0, \quad * \geq 3 \quad (h_2)$$

$$0 \rightarrow C_H^{*-1}(P, M^e) \oplus C_H^{*-1}(P, M^e) \rightarrow \text{cone}(-\beta^*, -\beta'^*) \rightarrow C_L^*(P, M) \rightarrow 0, \quad * \geq 3. \quad (h)$$

In these sequences i_2, i_3, i_4 and i_5 denote the injections on the corresponding summands, respectively. These exact sequences are obtained directly from the constructions of the cohomology complexes of the corresponding types of algebras.

Theorem 5.6. *We have the following exact sequences of cohomology vector spaces*

$$\begin{array}{ccccccc} H_{\text{AWB}^t}^2(P, M) & \longrightarrow & H_{\text{NP}^t}^2(P, M) & \longrightarrow & H_L^2(P, M) & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{\text{AWB}^t}^3(P, M) & \longrightarrow & H_{\text{NP}^t}^3(P, M) & \longrightarrow & H_L^3(P, M) & \longrightarrow & \dots \end{array} \quad (A_1)$$

where P is an \mathbf{NP}^l -algebra and M a representation of P .

$$\begin{array}{ccccccc} H_{\text{AWBr}}^2(P, M) & \longrightarrow & H_{\text{NPr}}^2(P, M) & \longrightarrow & H_L^2(P, M) & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{\text{AWBr}}^3(P, M) & \longrightarrow & H_{\text{NPr}}^3(P, M) & \longrightarrow & H_L^3(P, M) & \longrightarrow & \dots \end{array} \quad (A_2)$$

where P is an \mathbf{NP}^r -algebra and M a representation of P .

$$\begin{array}{ccccccc} H_{\text{AWB}^{lr}}^2(P, M) & \longrightarrow & H_{\text{NP}^{lr}}^2(P, M) & \longrightarrow & H_L^2(P, M) & \longrightarrow & \dots \\ & & & & \downarrow & & \\ & & & & H_L^3(P, M) & & \end{array} \quad (A)$$

where P is an \mathbf{NP}^{lr} -algebra and M a representation of P .

[illegible]

where P is an \mathbf{NP}^l -algebra and M a representation of P .

$$\begin{array}{c}
H^3(\text{cone}(-\beta'^*)) \longrightarrow H_{\text{NP}^r}^3(P, M) \longrightarrow H_H^3(P, M) \\
\curvearrowright \\
\longrightarrow H^4(\text{cone}(-\beta'^*)) \longrightarrow H_{\text{NP}^r}^4(P, M) \longrightarrow H_H^4(P, M) \longrightarrow \dots
\end{array} \quad (B_2)$$

where P is an NP^r -algebra and M a representation of P .

$$\begin{array}{c}
H^3(\text{cone}(-\beta^*, -\beta'^*)) \longrightarrow H_{\text{NP}^{lr}}^3(P, M) \longrightarrow H_H^3(P, M) \\
\curvearrowright \\
\longrightarrow H^4(\text{cone}(-\beta^*, -\beta'^*)) \longrightarrow H_{\text{NP}^{lr}}^4(P, M) \longrightarrow H_H^4(P, M) \longrightarrow \dots
\end{array} \quad (B)$$

where P is an NP^{lr} -algebra and M a representation of P .

$$\begin{array}{c}
H_H^2(P, M^e) \longrightarrow H_{\text{AWB}^r}^3(P, M) \longrightarrow H_H^3(P, M) \\
\curvearrowright \\
\longrightarrow H_H^3(P, M^e) \longrightarrow H_{\text{AWB}^r}^4(P, M) \longrightarrow H_H^4(P, M) \longrightarrow \dots
\end{array} \quad (C_{1,2})$$

where P is an AWB^r and M a representation of P . Analogous exact sequence we have for $H_{\text{AWB}^l}(P, M)$.

$$\begin{array}{c}
H_H^2(P, M^e) \longrightarrow H_{\text{AWB}^{lr}}^3(P, M) \longrightarrow H_{\text{AWB}^r}^3(P, M) \\
\curvearrowright \\
\longrightarrow H_H^3(P, M^e) \longrightarrow H_{\text{AWB}^{lr}}^4(P, M) \longrightarrow H_{\text{AWB}^r}^4(P, M) \longrightarrow \dots
\end{array} \quad (C, C')$$

where P is an AWB^{lr} and M a representation of P . Analogous exact sequence we have, where $H_{\text{AWB}^r}(P, M)$ is replaced by $H_{\text{AWB}^l}(P, M)$.

$$\begin{array}{c}
H_H^2(P, M^e) \longrightarrow H_{\text{NP}^r}^3(P, M) \longrightarrow H_H^3(P, M) \oplus H_L^3(P, M) \\
\curvearrowright \\
\longrightarrow H_H^3(P, M^e) \longrightarrow H_{\text{NP}^r}^4(P, M) \longrightarrow H_H^4(P, M) \oplus H_L^4(P, M) \longrightarrow \dots
\end{array} \quad (D_{1,2})$$

where P is an NP^r -algebra and M a representation of P . Analogously for $H_{\text{NP}^l}(P, M)$.

$$\begin{array}{c}
H_H^2(P, M^e) \longrightarrow H_{\text{NP}^{lr}}^3(P, M) \longrightarrow H_{\text{NP}^r}^3(P, M) \\
\curvearrowright \\
\longrightarrow H_H^3(P, M^e) \longrightarrow H_{\text{NP}^{lr}}^4(P, M) \longrightarrow H_{\text{NP}^r}^4(P, M) \longrightarrow \dots
\end{array} \quad (D, D')$$

where P is an NP^{lr} -algebra and M a representation of P . Analogous exact sequence we have, where $H_{\text{NP}^r}(P, M)$ is replaced by $H_{\text{NP}^l}(P, M)$.

$$\begin{array}{c}
H_H^2(P, M^e) \oplus H_H^2(P, M^e) \longrightarrow H_{\text{AWB}^{lr}}^3(P, M) \longrightarrow H_H^3(P, M) \\
\curvearrowright \\
\longrightarrow H_H^3(P, M^e) \oplus H_H^3(P, M^e) \longrightarrow H_{\text{AWB}^{lr}}^4(P, M) \longrightarrow H_H^4(P, M) \longrightarrow \dots
\end{array} \quad (E)$$

where P is an AWB^{lr} and M a representation of P .

$$\begin{array}{c}
H_H^2(P, M^e) \oplus H_H^2(P, M^e) \longrightarrow H_{\text{NP}^{lr}}^3(P, M) \longrightarrow H_H^3(P, M) \oplus H_L^3(P, M) \\
\curvearrowright \\
\longrightarrow H_H^3(P, M^e) \oplus H_H^3(P, M^e) \longrightarrow H_{\text{NP}^{lr}}^4(P, M) \longrightarrow H_H^4(P, M) \oplus H_L^4(P, M) \longrightarrow \dots
\end{array} \quad (F)$$

where P is an NP^{lr} -algebra and M a representation of P .

$$\begin{array}{c}
H_H^2(P, M^e) \longrightarrow H_{\text{AWB}^l}^3(P, M) \oplus H^3(\text{cone}(-\beta^*)) \longrightarrow H_{\text{NP}^l}^3(P, M) \\
\curvearrowright \\
\longrightarrow H_H^3(P, M^e) \longrightarrow H_{\text{AWB}^l}^4(P, M) \oplus H^4(\text{cone}(-\beta^*)) \longrightarrow H_{\text{NP}^l}^4(P, M) \longrightarrow \dots
\end{array} \quad (G_1)$$

where P is an NP^l -algebra and M a representation of P .

$$\begin{array}{c}
H_H^2(P, M^e) \longrightarrow H_{\text{AWB}^r}^3(P, M) \oplus H^3(\text{cone}(-\beta'^*)) \longrightarrow H_{\text{NP}^r}^3(P, M) \\
\curvearrowright \\
\longrightarrow H_H^3(P, M^e) \longrightarrow H_{\text{AWB}^r}^4(P, M) \oplus H^4(\text{cone}(-\beta'^*)) \longrightarrow H_{\text{NP}^r}^4(P, M) \longrightarrow \dots
\end{array} \quad (G_2)$$

where P is an NP^r -algebra and M a representation of P .

$$\begin{array}{c}
H_H^2(P, M^e) \oplus H_H^2(P, M^e) \longrightarrow H_{\text{AWB}^{lr}}^3(P, M) \oplus H^3(\text{cone}(-\beta^*, -\beta'^*)) \longrightarrow \\
\longrightarrow H_{\text{NP}^{lr}}^3(P, M) \longrightarrow H_H^3(P, M^e) \oplus H_H^3(P, M^e) \longrightarrow \\
\longrightarrow H_{\text{AWB}^{lr}}^4(P, M) \oplus H^4(\text{cone}(-\beta^*, -\beta'^*)) \longrightarrow H_{\text{NP}^{lr}}^4(P, M) \longrightarrow \dots
\end{array} \quad (G)$$

where P is an NP^{lr} -algebra and M a representation of P .

$$\begin{array}{c}
H_H^2(P, M^e) \longrightarrow H^3(\text{cone}(-\beta^*)) \longrightarrow H_L^3(P, M) \\
\curvearrowright \\
\longrightarrow H_H^3(P, M^e) \longrightarrow H^4(\text{cone}(-\beta^*)) \longrightarrow H_L^4(P, M) \longrightarrow \dots
\end{array} \quad (H_{1,2})$$

where P is an NP^l -algebra and M a representation of P . Analogous exact sequence we have for the cohomologies of the $\text{cone}(-\beta'^*)$ and for an NP^r -algebra P .

$$\begin{array}{c}
H_H^2(P, M^e) \oplus H_H^2(P, M^e) \longrightarrow H^3(\text{cone}(-\beta^*, -\beta'^*)) \longrightarrow H_L^3(P, M) \longrightarrow \\
\longrightarrow H_H^3(P, M^e) \oplus H_H^3(P, M^e) \longrightarrow H^4(\text{cone}(-\beta^*, -\beta'^*)) \longrightarrow H_L^4(P, M) \longrightarrow \dots
\end{array} \quad (H)$$

for any NP^{lr} -algebra P and a representation M of P .

Proof. These exact sequences are obtained directly from the corresponding short exact sequences of the cohomology complexes. \square

Recall that *Hochschild cohomological dimension* $\text{c.dim}_H P$ of an associative algebra P is defined as the greatest natural number n , for which there exists a P - P -bimodule S with $H_H^n(P, S) \neq 0$. The analogous meaning will have *Leibniz cohomological dimension* of a Leibniz algebra P , *AWB cohomological dimension* of an algebra $P \in \text{AWB}$ and *NP cohomological dimension* of an NP-algebra P , denoted as $\text{c.dim}_L P$, $\text{c.dim}_{\text{AWB}} P$ and $\text{c.dim}_{\text{NP}} P$, respectively.

Corollary 5.7. *Let P be a free NP^r -algebra (resp. NP^l -algebra) and M be a representation of P . Then we have*

- (i) $H_{\text{AWB}^r}^n(P, M) = 0, n \geq 3$ (resp. $H_{\text{AWB}^l}^n(P, M) = 0, n \geq 3$), where P is the underlying AWB^r (resp. AWB^l) of the given algebra P ;
- (ii) $H^n(\text{cone}(-\beta'^*)) = 0$ (resp. $H^n(\text{cone}(-\beta^*)) = 0$), $n \geq 3$.

Proof. (i) Since P is a free NP^r -algebra, by Corollary 5.3 (ii) $H_{\text{NP}^r}^n(P, M) = 0, n \geq 3$. By Proposition 3.3 the underlying Leibniz algebra is also free, from which follows the well-known fact that $H_L^n(P, N) = 0, n \geq 2$, for any representation N of P in the category of Leibniz algebras **Leib**, i.e., $\text{c.dim}_L P \leq 1$. Since M is a representation of P in the category of NP^r -algebras, it follows that it is a representation of P in **Leib** as well, P considered as the underlying Leibniz algebra. Now the result follows from long exact sequence (A_2) in Theorem 5.6. Analogously for $P \in \text{NP}^l$, where we apply (A_1) .

(ii) We apply again Proposition 3.3 and conclude that the underlying associative algebra of P is free, from which we have that $\text{c.dim}_H P \leq 1$. The result follows from the statement (i) of this Corollary, Corollary 5.3 (ii) and the exact sequence (G_2) . Note that the statement (ii) for $n \geq 4$ follows from (B_2) as well. Analogously we obtain the equality $H^n(\text{cone}(-\beta^*)) = 0$, where we apply exact sequence (G_1) in Theorem 5.6. \square

Corollary 5.8. *Let P be an AWB. If $\text{c.dim}_H P \leq n, n \geq 1$, where P is the underlying associative algebra, then*

$$\text{c.dim}_{\text{AWB}} P \leq n + 1.$$

Proof. Let P be an AWB^r or an AWB^l . The results follow from exact sequences $(C_{1,2})$ in Theorem 5.6. Let P be a left-right AWB. Applying the result for AWB^r (or AWB^l) for the underlying algebra P as an AWB^r (resp. as an AWB^l), the result follows from exact sequences (C, C') in Theorem 5.6. \square

Corollary 5.9. *Let P be a NP^{lr} -algebra and $\text{c.dim}_H P \leq n, n \geq 2$. If M is a representation of P , then we have:*

- (i) $H_{\text{NP}^{lr}}^{k+1}(P, M) \approx H^{k+1}(\text{cone}(-\beta^*, -\beta'^*)), H_{\text{NP}^l}^{k+1}(P, M) \approx H^{k+1}(\text{cone}(-\beta^*)), H_{\text{NP}^r}^{k+1}(P, M) \approx H^{k+1}(\text{cone}(-\beta'^*)), k > n$, where in the last two isomorphisms P denotes the underlying NP^l and NP^r -algebras of the given NP^{lr} -algebra P , respectively;
- (ii) $H_{\text{NP}^{lr}}^{k+1}(P, M) \approx H_{\text{NP}^r}^{k+1}(P, M) \approx H_{\text{NP}^l}^{k+1}(P, M), k > n$, where P in the last two right terms denotes the underlying NP^l and NP^r -algebras of the given algebra P , respectively;

- (iii) $H_{\text{NP}^{\text{tr}}}^{k+1}(P, M) \approx H_L^{k+1}(P, M)$, $k > n$, where on the right side P denotes the underlying Leibniz algebra of the given algebra P .

Proof. (i) Follows from exact sequences $(B_1), (B_2)$ and (B) in Theorem 5.6. Analogously, for the proofs of (ii) and (iii) we apply exact sequences (D, D') and (F) , respectively. Note that (iii) can be obtained as well by application of statement (i) of this corollary and exact sequence (H) . \square

The below stated corollaries are proved due to analogous arguments, therefore the proofs are left to the reader.

Corollary 5.10. *Let P be an NP-algebra and M be a representation of P . If $\text{c. dim}_H P \leq n$ and $\text{c. dim}_L P \leq n$, $n \geq 2$, where P denotes the underlying associative and Leibniz algebras, respectively, then we have:*

- (i) $\text{c. dim}_{\text{NP}} P \leq n + 1$;
- (ii) $H^{k+1}(\text{cone}(-\beta^*)) = H^{k+1}(\text{cone}(-\beta'^*)) = H^{k+1}(\text{cone}(-\beta^*, -\beta'^*)) = 0$, $k > n$.

Corollary 5.11. *Let P be an NP-algebra and M be a representation of P . If $\text{c. dim}_L P \leq n$, where P is the underlying Leibniz algebra, then we have:*

- (i) $H_{\text{NP}}^{k+1}(P, M) \approx H_{\text{AWB}}^{k+1}(P, M)$, $k > n$;
- (ii) $H^{k+1}(\text{cone}(-\beta^*)) \approx H^{k+1}(\text{cone}(-\beta'^*)) \approx H^{k+1}(\text{cone}(-\beta^*, -\beta'^*)) \approx H_H^k(P, M^e)$, $k > n$.

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REFERENCES

- [1] Borceux, F., Janelidze, G., Kelly, G.M., Internal object actions, *Comment. Math. Univ. Carolinae*, 2005, **46**(2), 235–255
- [2] Borceux, F., Janelidze, G., Kelly, G.M., On the representability of actions in a semi-abelian category, *Theory Appl. Categ.*, 2005, **14**(11), 244–286
- [3] Bourn, D., Janelidze, G., Centralizers in action accessible categories, *Cah. Topol. Géom. Différ. Catég.*, 2009, **50**(3), 211–232
- [4] Casas, J.M., Datuashvili, T., Noncommutative Leibniz–Poisson algebras, *Comm. Algebra*, 2006, **34**(7), 2507–2530
- [5] Casas, J.M., Datuashvili, T., Ladra, M., Actors in categories of interest, arXiv:math/0702574v2
- [6] Casas, J.M., Datuashvili, T., Ladra, M., Actor of a precrossed module, *Comm. Algebra*, 2009, **37**(12), 4516–4541
- [7] Casas, J.M., Datuashvili, T., Ladra, M., Actor of an alternative algebra, arXiv:math.RA/0910.0550v1
- [8] Casas, J.M., Datuashvili, T., Ladra, M., Universal strict general actors and actors in categories of interest, *Appl. Categ. Structures*, 2010, **18**(1), 85–114
- [9] Casas, J.M., Pirashvili, T., Algebras with bracket, *Manuscripta Math.*, 2006, **119**(1), 1–15
- [10] Cornish, W.H., Amalgamating commutative regular rings, *Bull. Aust. Math. Soc.*, 1977, **16**, 1–13

- [11] Datuashvili, T., Cohomologically trivial internal categories in categories of groups with operations, *Appl. Categ. Structures*, 1995, **3**(3), 221–237
- [12] Dotsenko, V., Khoroshkin, A. Gröbner bases for operads, *Duke Math. J.*, 2010, **153**(2), 363–396
- [13] Fresse, B., Homologie de Quillen pour les algèbres de Poisson, *C. R. Acad. Sci. Paris Sér. I Math.*, 1998, **326**(9), 1053–1058
- [14] Fresse, B., Théorie des opérades de Koszul et homologie des algèbres de Poisson, *Ann. Math. Blaise Pascal*, 2006, **13**(2), 237–312
- [15] Higgins, P.J., Groups with multiple operators, *Proc. London Math. Soc. (3)*, 1956, **6**, 366–416
- [16] Hochschild, G., Cohomology and representations of associative algebras, *Duke Math. J.*, 1947, **14**, 921–948
- [17] Hoffbeck, E., Poincaré-Birkhoff-Witt criterion for Koszul operads, *Manuscripta Math.*, 2010, **131**(1-2), 87–110
- [18] Huebschmann, J., Poisson cohomology and quantization, *J. Reine Angew. Math.*, 1990, **408**, 57–113
- [19] Kanatchikov, I.V., On field-theoretic generalizations of a Poisson algebra, Quantizations, deformations and coherent states, Białowieża, 1996, *Rep. Math. Phys.*, 1997, **40**(2), 225–234
- [20] Kubo, F., Finite-dimensional non-commutative Poisson algebras, *J. Pure Appl. Algebra*, 1996, **113**(3), 307–314
- [21] Kubo, F., Non-commutative Poisson algebra structures on affine Kac-Moody algebras, *J. Pure Appl. Algebra*, 1998, **126**(1-3), 267–286
- [22] Loday, J.-L., *Cyclic homology*, Appendix E by M. O. Ronco, Grundlehren der Mathematischen Wissenschaften, 301, Springer-Verlag, Berlin, 1992
- [23] Loday, J.-L., Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *Enseign. Math. (2)*, 1993, **39**(3-4), 269–293
- [24] Loday, J.-L., Algèbres ayant deux opérations associatives (digèbres), *C. R. Acad. Sci. Paris Sér. I Math.*, 1995, **321**, 141–146
- [25] Loday, J.-L., *Dialgebras. Dialgebras and related operads*, 7–66, Lecture Notes in Math., 1763, Springer, Berlin, 2001
- [26] Loday, J.-L., Pirashvili, T., Universal enveloping algebras of Leibniz algebras and (co)homology, *Math. Ann.*, 1993, **296**(1), 139–158
- [27] Loday, J.-L., Ronco, M.O., Trialgebras and families of polytopes, Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-theory, 369–398, Contemp. Math., 346, Amer. Math. Soc., Providence, R.I., 2004
- [28] Loday, J.-L., Vallette, B., Algebraic operads, Grundlehren der Mathematischen Wissenschaften, 346, Springer, Heidelberg, 2012
- [29] Montoli, A., Action accessibility for categories of interest, *Theory Appl. Categ.*, 2010, **23**(1), 7–21
- [30] Orzech, G., Obstruction theory in algebraic categories I and II, *J. Pure Appl. Algebra*, 1972, **2**, 287–314 and 315–340
- [31] Porter, T., Extensions, crossed modules and internal categories in categories of groups with operations, *Proc. Edinburgh Math. Soc. (2)*, 1987, **30**(3), 373–381
- [32] Quillen, D., On the (co-)homology of commutative rings, Applications of Categorical Algebra, 65–87, Proc. Sympos. Pure Math., Vol. XVII, New York, 1968, Amer. Math. Soc., Providence, R.I., 1970
- [33] Tong, J., Jin, Q. Q., Non-commutative Poisson algebra structures on the Lie algebra $\mathfrak{so}_n(\widehat{C}_{\mathbb{Q}})$, *Algebra Colloq.*, 2007, **14**(3), 521–536
- [34] Xu, P., Noncommutative Poisson algebras, *Amer. J. Math.*, 1994, **116**(1), 101–125